Lecture 38

Thursday, December 2, 2021 9:33 PM

* Prayer
* Spiritual thought

How to find the suitable coefficients to satisfy the initial conditions?

$$Y' = AY, \quad Y(t_0) = Y_0 \text{ given}$$

$$Y = c_1 Y^{(1)} + \dots + c_n Y^{(n)}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ Y^{(1)} & \cdots & Y^{(n)} \\ | & & | \end{bmatrix}}_{\Phi\text{ - fundamental matrix}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Substitute $t = t_0$:

$$Y_0 = \Phi(t_0) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightsquigarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi(t_0)^{-1} Y_0 .$$

It would be desirable if $\Phi(t_0) = I_n$.

Ex

$$Y' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} Y, \quad Y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solve for Y.

$\lambda_1 = i, \quad \lambda_2 = -i$

$$v_1 = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

$$Y^{(1)} = v_1 e^{\lambda_1 t} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} e^{it} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} (\cos t + i \sin t)$$

$$= \begin{bmatrix} 2\cos t + \sin t - i\cos t + 2i \sin t \\ \cos t + i \sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2\cos t + \sin t \\ \cos t \end{bmatrix}}_{u} + i \underbrace{\begin{bmatrix} -\cos t + 2\sin t \\ \sin t \end{bmatrix}}_{v}$$

General solution: $Y = c_1 u + c_2 v = \underbrace{\begin{bmatrix} 2\cos t + \sin t & -\cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}}_{\text{fundamental matrix } \Phi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Find the fundamental matrix such that $\Psi(0) = I_2$.

$$\Psi(t) = \begin{bmatrix} | & | \\ d_1 Y^{(1)} + d_2 Y^{(2)} & e_1 Y^{(1)} + e_2 Y^{(2)} \\ | & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & | \\ Y^{(1)} & Y^{(2)} \\ | & | \end{bmatrix}}_{\phi(t)} \begin{bmatrix} d_1 & e_1 \\ d_2 & e_2 \end{bmatrix}$$

At $t = 0$:

$$I_2 = \Psi(0) = \phi(0) \begin{bmatrix} d_1 & e_1 \\ d_2 & e_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 & e_1 \\ d_2 & e_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} d_1 & e_1 \\ d_2 & e_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} Y^{(1)} & Y^{(2)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2\cos t + \sin t & -\cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - 2\sin t & 3\sin t \\ -\sin t & \cos t + 2\sin t \end{bmatrix}$$

$$Y' = AY$$

- A is diagonalizable: good $\to$ $Y = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$.
- A is not diagonalizable: needs more work.

Say, λ_1 is a double root $(\lambda_1 = \lambda_2)$ and there's only one eigenvector

$$Y^{(1)} = v_1 e^{\lambda_1 t}$$

$$Y^{(2)} = (tv_1 + w)e^{\lambda_1 t} \quad \text{— choose } w \text{ suitably.}$$

For $Y^{(2)}$ to be a solution, we need $Y^{(2)\prime} = AY^{(2)}$

$$\begin{cases} Y^{(2)\prime} = v_1 e^{\lambda_1 t} + \lambda_1 t v_1 e^{\lambda_1 t} + \lambda_1 w e^{\lambda_1 t} \\ AY^{(2)} = A(tv_1 + w)e^{\lambda_1 t} = tAve^{\lambda_1 t} + Awe^{\lambda_1 t} \\ \qquad = t\lambda_1 v e^{\lambda_1 t} + Awe^{\lambda_1 t} \end{cases}$$

$\rightsquigarrow$ need $Aw = \lambda_1 w + v_1$

$\rightsquigarrow$ $(A - \lambda_1 I_n)w = v_1$

$\underline{Ex}$

$$Y' = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{A} Y, \quad Y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda = 2$ is a double root

Only one eigenvector $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\rightsquigarrow Y^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$.

$$Y^{(2)} = \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + w \right) e^{2t}.$$

How to find w?

$$(A - \lambda I_2) w = v \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow b = 1$$

(choose $a = 0$)

$$w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow Y^{(2)} = \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

Use Mathematica: improper node (hybrid between a node and a spiral point).